International Journal of Mathematics and Physical Sciences Research ISSN 2348-5736 (Online)
Vol. 14, Issue 1, pp: (76-90), Month: April 2026 - September 2026, Available at: www.researchpublish.com

Modified Gravity as an Alternative to Dark Energy: A Survey of Curvature, Torsion, Non-metricity and Matter Geometry Coupling Theories

S. R. Bhoyar[1], A. D. Wankhade[1*]

[1]Department of Mathematics, Phulsing Naik Mahavidyalaya, Pusad -445216, Affiliated to Sant Gadge Baba Amravati University, Amravati 444602, Maharashtra, India.

DOI: https://doi.org/10.5281/zenodo.21255139

Published Date: 07-July-2026

Abstract: **The observed late-time acceleration of the Universe remains one of the deepest open problems in fundamental physics. Within general relativity it demands a dark energy component most economically a cosmological constant Λ whose observationally inferred value is separated from naive quantum field theoretic expectations by tens of orders of magnitude. Modified theories of gravity offer an alternative: cosmic acceleration as a manifestation of infrared modifications of the gravitational sector itself. This review surveys the principal geometric routes to such modifications, organized by the geometric trinity of gravity: curvature-based theories ($f(R)$, scalar–tensor and Horndeski, Gauss–Bonnet extensions), torsion-based theories ($f(T)$ and generalizations), and non-metricity-based theories ($f(Q)$ and generalizations), together with the increasingly studied class of matter–geometry coupling theories, $f(R,T)$, $f(R,L_m)$, $f(Q,T)$, $f(Q,L_m)$ and energy–momentum-squared gravity. We apply three viability filters uniformly across all classes: (i) local and astrophysical constraints, foremost the multimessenger bound $|c_T/c-1| \lesssim 10^{-15}$ from GW170817; (ii) consistency with late-time cosmological observations and the status of the H_0 and S_8 tensions; and (iii) thermodynamic viability, i.e. the generalized second law and non-negative entropy production at the apparent horizon, a criterion that can discriminate between models degenerate at the background level and that acquires particular significance in matter–geometry coupling theories, where non-conservation of the energy–momentum tensor is naturally interpreted as irreversible gravitationally induced particle creation. We summarize the theoretical pathologies afflicting each class, delineate the surviving theory space, and outline the discriminating power of forthcoming surveys and gravitational-wave observations.**

Keywords: **modified gravity; dark energy; $f(R)$ gravity; $f(Q)$ gravity; matter–geometry coupling; generalized second law; cosmic acceleration.**

1. INTRODUCTION

The discovery of the accelerated expansion of the Universe from Type Ia supernova observations [1, 2] transformed the cosmological constant from a historical curiosity into the central puzzle of fundamental physics. Within the standard ΛCDM model, acceleration is driven by a constant vacuum energy density $\rho_\Lambda \simeq 10^{-47}$ GeV^4, a value that quantum field theoretic estimates of vacuum energy exceed by some 60–120 orders of magnitude depending on the cutoff adopted [3]. To this old cosmological constant problem one adds the coincidence problem why should $\rho_\Lambda \sim \rho_m$ precisely today and, more recently, a set of empirical strains on ΛCDM itself: the persistent discrepancy, exceeding 5σ in some analyses, between local distance-ladder determinations of the Hubble constant [4] and the value inferred from the cosmic microwave background under ΛCDM [5]; the milder S_8 amplitude tension between weak-lensing and CMB determinations of structure growth [6, 7, 8]; and indications from the DESI baryon acoustic oscillation programme that the dark energy equation of state may evolve with redshift [9].

Broadly, two responses are available. One retains general relativity (GR) and populates the energy budget with a dynamical dark energy component quintessence, k-essence, and holographic constructions with their many generalizations [10, 11, 12, 14, 15]. The other modifies the gravitational dynamics in the infrared, so that acceleration is a geometric effect requiring no exotic fluid. It is the second programme that concerns us here. The logical space of such modifications is sharply constrained by Lovelock's theorem [16]: in four dimensions, the only divergence-free, symmetric, second-order Euler–Lagrange expression built from the metric alone is the Einstein tensor plus a cosmological term. Every modified gravity theory must therefore break at least one Lovelock assumption by adding fields, extra dimensions, higher derivatives, or by changing the underlying geometry [17, 18, 19].

A particularly transparent organizing principle is the geometric trinity of gravity. GR attributes gravitation to the curvature R of the Levi-Civita connection; the teleparallel equivalent of GR (TEGR) attributes it to the torsion scalar T of a curvature-free connection; and the symmetric teleparallel equivalent (STEGR) attributes it to the non-metricity scalar Q of a connection that is both flat and torsion-free [21, 22]. The three formulations are dynamically equivalent their Lagrangians differ by boundary terms but their nonlinear extensions $f(R)$, $f(T)$ and $f(Q)$ are inequivalent theories with distinct degrees of freedom, distinct perturbative behaviour, and distinct pathologies. A fourth direction, orthogonal to the trinity, generalizes the matter coupling: theories such as $f(R, L_m)$ [23], $f(R,T)$ [24], $f(Q,T)$ [25], $f(Q, L_m)$ [26] and energy–momentum-squared gravity $f(R, \mathbf{T}^2)$ [27, 28, 29] couple geometric invariants non-minimally to the matter Lagrangian or to the energy–momentum tensor, generically producing non-conservation of $T_{\mu\nu}$ and admitting a thermodynamic interpretation in terms of irreversible particle creation [31, 33].

Several authoritative reviews cover portions of this landscape: dark energy generally [13, 10, 11], the full modified-gravity taxonomy [17, 20], $f(R)$ gravity [35, 36, 37], inflation and dark energy in extended gravity [38, 39], teleparallel theories [40, 41], $f(Q)$ gravity [42], cosmological tests of gravity [43], and the observational programme of the CANTATA network [44]. The purpose of the present survey is complementary in three respects. First, we treat the matter–geometry coupling theories which have generated a large literature over the past decade but remain peripheral in the reviews above on the same footing as the trinity extensions. Second, we apply the post-GW170817 multimessenger constraint uniformly across all classes, since the bound $|c_T/c - 1| \lesssim 10^{-15}$ [45, 46] eliminated large regions of theory space at a stroke [47, 48, 49]. Third, we elevate thermodynamic viability validity of the generalized second law (GSL) at the apparent horizon and non-negativity of entropy production to a systematic selection criterion, in the spirit of the gravity–thermodynamics correspondence [51, 52, 53]. Background expansion histories in different theories are often observationally degenerate; their entropy budgets frequently are not.

The review is organized as follows. Section 2 fixes conventions and summarizes the geometric trinity. Sections 3, 4 and 5 survey curvature-, torsion- and non-metricity-based theories respectively. Section 6 treats matter–geometry coupling theories. Section 7 confronts all classes with local, astrophysical and cosmological data; Section 8 assesses their capacity to address the cosmological tensions. Section 9 develops the thermodynamic viability criterion. Section 10 collects the theoretical pathologies, and Section 12 summarizes the surviving theory space and future prospects. We use metric signature $(-,+,+,+)$, natural units $c = \hbar = 1$, and $\kappa^2 = 8\pi G$ unless stated otherwise.

2. GEOMETRIC FOUNDATIONS: THE TRINITY OF GRAVITY

Figure 1 illustrates the organizational structure of modified gravity theories surveyed in this review. General relativity sits at the apex; the three geometric equivalents give rise to the trinity of nonlinear extensions; and matter–geometry coupling theories form a fourth, orthogonal direction.

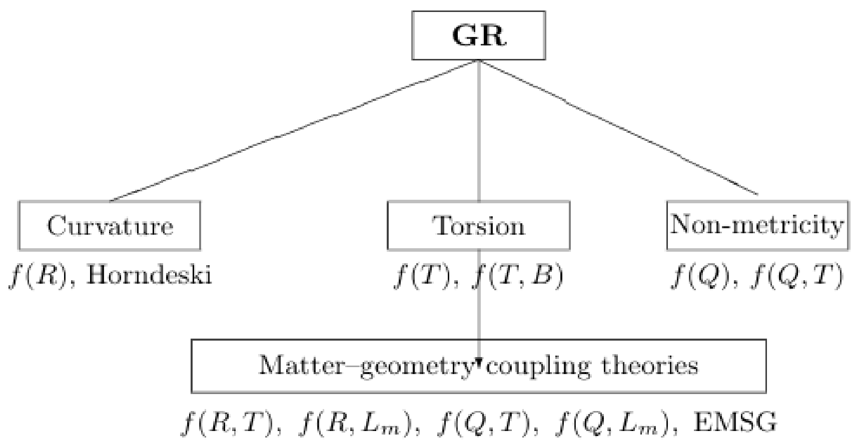

Figure 1: Organizational map of the modified gravity theories reviewed here. GR sits at the apex of the geometric trinity (curvature, torsion, non-metricity). Nonlinear extensions of each equivalent give distinct propagating degrees of freedom and pathologies. Matter–geometry coupling theories form a fourth direction orthogonal to the trinity, sharing the property $\nabla_\mu T_{\mu\nu} \neq 0$.

The most general affine connection decomposes as

$$\Gamma^\lambda{}_{\mu\nu} = \left\{{\lambda \atop \mu\nu}\right\} + K^\lambda{}_{\mu\nu} + L^\lambda{}_{\mu\nu}, \tag{1}$$

where $\left\{{\lambda \atop \mu\nu}\right\}$ is the Levi-Civita connection, $K^\lambda{}_{\mu\nu}$ is the contortion tensor built from the torsion $T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}$, and $L^\lambda{}_{\mu\nu}$ is the disformation tensor built from the non-metricity $Q_{\lambda\mu\nu} = \nabla_\lambda g_{\mu\nu}$. Setting torsion and non-metricity to zero yields Riemannian geometry and GR, with action

$$S_{\rm GR} = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,R + S_m. \tag{2}$$

Alternatively, one may impose vanishing curvature and non-metricity, encoding gravity in the torsion scalar T (TEGR), or vanishing curvature and torsion, encoding it in the non-metricity scalar

$$Q = -g^{\mu\nu}\left(L^\alpha{}_{\beta\mu}L^\beta{}_{\nu\alpha} - L^\alpha{}_{\beta\alpha}L^\beta{}_{\mu\nu}\right) \tag{3}$$

(STEGR). The three scalars are related by total derivatives, $R = -T + B_T = Q + B_Q$, where B_T and B_Q are boundary terms [22], so the linear theories coincide dynamically while the nonlinear extensions $f(R)$, $f(T)$ and $f(Q)$ do not. In particular, $f(R)$ propagates one extra scalar degree of freedom and yields fourth-order metric field equations; $f(T)$ and $f(Q)$ retain second-order equations at the price of breaking local Lorentz invariance in the pure-tetrad formulation ($f(T)$) or introducing a dynamical connection sector with delicate gauge structure ($f(Q)$) [40, 42].

Throughout we specialize, unless otherwise stated, to the spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t)\,\delta_{ij}\,dx^i dx^j, \tag{4}$$

for which, in the conventions adopted here, $T = 6H^2$ and, in the coincident gauge, $Q = 6H^2$, with $H = \dot{a}/a$ the Hubble parameter. We caution the reader that sign conventions for T and Q vary across the literature (definitions with $T = -6H^2$ are equally common), and all Friedmann equations below are written consistently with the convention just declared.

Figure 2. Timeline of key developments in modified gravity and cosmological observations. The post-2017 period is dominated by the GW170817-induced restructuring of theory space and by the entry of precision large-scale structure surveys.

Year	Development
1915	General relativity (Einstein)
1961	Brans–Dicke scalar–tensor theory
1970	Lovelock's theorem
1974	Horndeski's general scalar–tensor Lagrangian
1980	Starobinsky R^2 inflation
1989	Prigogine particle-creation cosmology
1998	Discovery of cosmic acceleration (Riess, Perlmutter)
2003	$f(R)$ dark energy models
2007	Hu–Sawicki viable $f(R)$ model
2010	$f(R, L_m)$ and $f(R, T)$ matter-coupling theories
2011	$f(T)$ teleparallel dark energy
2014	Energy–momentum-squared gravity (EMSG)
2017	GW170817: $c_T = c$, standard siren H_0
2018	Symmetric teleparallel $f(Q)$ gravity
2019	$f(Q, T)$ gravity
2023	PTA nanohertz GW background; DESI hints; $f(Q, L_m)$
2024–	DESI, Euclid, Rubin: precision tests of modified gravity

3. CURVATURE-BASED THEORIES

3.1 $f(R)$ gravity

The oldest and most thoroughly studied extension replaces the Einstein–Hilbert Lagrangian by an arbitrary function of the Ricci scalar [60, 61, 35, 36],

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,f(R) + S_m, \tag{5}$$

with metric field equations

$$f_R R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right) f_R = \kappa^2 T_{\mu\nu}, \qquad f_R \equiv \frac{\partial f}{\partial R}. \tag{6}$$

On the FLRW background the Friedmann equations become

$$3 f_R H^2 = \kappa^2\rho + \frac{1}{2}(f_R R - f) - 3H\dot{f}_R, \tag{7}$$

$$-2 f_R \dot{H} = \kappa^2(\rho + p) + \ddot{f}_R - H\dot{f}_R, \tag{8}$$

so that the departure from GR acts as an effective dark-energy fluid with

$$\rho_{\rm eff} = \frac{1}{\kappa^2}\left[\frac{1}{2}(f_R R - f) - 3H\dot{f}_R + 3(1 - f_R)H^2\right], \tag{9}$$

and analogously for the effective pressure. The theory is dynamically equivalent to a scalar–tensor theory in which the scalaron $\phi \sim f_R$ carries the extra degree of freedom. Viability requires $f_R > 0$ (the graviton is not a ghost), $f_{RR} > 0$ (no tachyonic scalaron instability, the Dolgov–Kawasaki condition), recovery of GR at high curvature, and a chameleon-screened scalaron to pass solar-system tests [36, 66, 68, 69]. Historically, the first dark-energy model of this class, $f = R - \mu^4/R$, failed precisely here: the light scalaron mediates an unscreened fifth force incompatible with solar-system data [64, 65]. The surviving representatives are of the Hu–Sawicki [62] and Starobinsky disappearing-cosmological-constant [63] type, e.g.

$$f(R) = R - m^2\frac{c_1\left(R/m^2\right)^n}{c_2(R/m^2)^n + 1}, \tag{10}$$

which reduce to ΛCDM in the limit $|f_{R0}| \equiv |f_R(R_0) - 1| \to 0$ and remain observationally viable only for small departures, with astrophysical screening tests pushing the bound to roughly $|f_{R0}| \lesssim 10^{-6}$ for the $n = 1$ Hu–Sawicki model [44, 43]. Because tensor perturbations propagate at exactly the speed of light in $f(R)$ gravity, the GW170817 bound is passed automatically. The consequence is that distinguishable phenomenology now resides almost entirely in structure growth: below the scalaron Compton wavelength the effective Newton constant is enhanced by a factor up to $4/3$, modifying the growth rate and the lensing potential in a scale-dependent way that Euclid- and Rubin-class surveys are designed to detect [67, 43].

3.2 Scalar–tensor and Horndeski theories

The most general four-dimensional scalar–tensor theory with second-order field equations is the Horndeski Lagrangian [71], encompassing Brans–Dicke theory [72], quintessence [73], k-essence, kinetic gravity braiding and covariant Galileons [74, 75] as special cases. Its role as a dark energy framework was drastically curtailed by GW170817: demanding $c_T = c$ without fine tuning eliminates the quartic and quintic sectors with derivative couplings to curvature, leaving essentially a conformally coupled sector $G_4(\phi)R$ together with the $G_2(\phi, X)$ and $G_3(\phi, X)$ terms [47, 48, 49, 76]. Self-accelerating covariant Galileon models are additionally disfavoured by their integrated Sachs–Wolfe signature and by combined background data. Degenerate higher-order (DHOST) extensions evade the Ostrogradsky ghost by construction and admit a surviving subfamily with luminal gravitational waves, but these remain tightly constrained by the requirement that gravitational waves neither decay into dark energy fluctuations nor induce instabilities in it [50]. The braneworld DGP model [77] deserves mention as the historically influential prototype of self-acceleration from infrared-modified gravity: its self-accelerating branch contains a ghost and is disfavoured by expansion and growth data, but it bequeathed the Vainshtein screening mechanism [78] that underlies much of the surviving scalar–tensor theory space.

3.3 Gauss–Bonnet extensions: $f(\mathcal{G})$ and $f(R,\mathcal{G})$

With the Gauss–Bonnet invariant $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ topological in four dimensions, non-trivial dynamics arises only through nonlinear or scalar-coupled dependence, $f(\mathcal{G})$, $f(R,\mathcal{G})$ or $\xi(\phi)\mathcal{G}$ [79, 38, 39]. Such models can realize late-time acceleration and unified inflation–dark-energy scenarios, but they face stringent conditions from ghost avoidance and, crucially, from the propagation speed of tensor modes, since generic Gauss–Bonnet couplings modify c_T; after GW170817 the scalar–Gauss–Bonnet coupling relevant at late times is forced to be extremely small unless protected by special structure [47, 80]. We include this class chiefly for completeness and because holographic dark energy embeddings on Gauss–Bonnet backgrounds remain an active model-building arena.

4. TORSION-BASED THEORIES

In $f(T)$ gravity [82, 83, 84, 40] the action is built from the torsion scalar of the Weitzenböck connection,

$$S = \frac{1}{2\kappa^2}\int d^4x \; e\, f(T) + S_m, \qquad e = \det\left(e^A{}_\mu\right), \tag{11}$$

where $e^A{}_\mu$ is the tetrad field. For flat FLRW with $T = 6H^2$ in our conventions, the Friedmann equations read

$$12H^2 f_T - f = -2\kappa^2\rho, \tag{12}$$

$$48H^2 f_{TT}\dot{H} + f_T\left(12H^2 + 4\dot{H}\right) - f = 2\kappa^2 p, \tag{13}$$

with $f(T) = T$ recovering GR. Because the field equations are second order, $f(T)$ gravity evades the scalaron machinery of $f(R)$; the widely used power-law model $f(T) = T + \alpha\, T^b$ interpolates smoothly to ΛCDM as $b \to 0$ and fits supernova, BAO and cosmic-chronometer data at a level statistically comparable to ΛCDM [40, 41]. Tensor modes propagate at $c_T = c$, so the multimessenger constraint is passed automatically.

The principal concerns are theoretical rather than observational. In the pure-tetrad formulation the theory is not locally Lorentz invariant [85], so that different tetrads representing the same metric yield inequivalent dynamics; the covariant formulation restores invariance at the cost of introducing a flat spin connection that must be determined consistently [87]. The number of propagating degrees of freedom has been debated for over a decade, with Hamiltonian analyses variously finding one, three or five extra modes depending on the branch [86, 41], and cosmological perturbation theory around FLRW backgrounds exhibits strong-coupling behaviour: modes that are absent at linear order reappear nonlinearly, undermining the predictivity of linear perturbation theory [41]. Extensions such as $f(T,B)$ gravity, which reinstates the boundary term B_T connecting T to R, and $f(T,T_{\mathcal{G}})$ gravity with the teleparallel Gauss–Bonnet invariant, interpolate between the $f(T)$ and $f(R)$ limits and inherit pathologies of both parents in a parameter-dependent way [88, 41].

5. NON-METRICITY-BASED THEORIES

Symmetric teleparallel gravity and its extension

$$S = \frac{1}{2\kappa^2}\int d^4x \sqrt{-g}\, f(Q) + S_m \tag{14}$$

have become, since the foundational works [21, 89], arguably the most active model-building framework in late-time cosmology; see [42] for a comprehensive review. In the coincident gauge on flat FLRW, where the connection is trivialized and $Q = 6H^2$, the Friedmann equations take the form

$$6f_Q H^2 - \frac{1}{2}f = \kappa^2\rho, \tag{15}$$

$$\left(12H^2 f_{QQ} + f_Q\right)\dot{H} = -\frac{1}{2}\kappa^2(\rho + p), \tag{16}$$

so that $f(Q) = Q$ recovers GR exactly, while $f(Q) = Q + 2\Lambda$ reproduces ΛCDM. Remarkably, the family $f(Q) = Q + \alpha\sqrt{Q}$ leaves the background dynamics of GR untouched the square-root term drops out of Eq. (15) so that its signatures are purely perturbative, a vivid illustration of the general theme that background fits alone cannot select among these theories. Power-law and exponential models, $f(Q) = Q + \alpha Q^n$ and $f(Q) = Q\, e^{\lambda Q_0/Q}$, produce viable accelerating histories constrained against supernova, cosmic-chronometer and BAO data [90, 91], and several $f(Q)$ models predict a weakened effective gravitational coupling for perturbations at late times, with a correspondingly suppressed $f\sigma_8$, a feature of direct relevance to the S_8 tension [92, 93].

Two structural caveats now dominate the theoretical discussion. First, the coincident gauge is not always compatible with a given symmetry-reduced metric ansatz; for spatially curved FLRW and for static spherical symmetry there exist inequivalent connection branches yielding genuinely different dynamics, a subtlety frequently ignored in the model-building literature [42, 94]. Second, perturbative analyses indicate that generic $f(Q)$ theories suffer from ghostly or strongly coupled sectors around maximally symmetric and FLRW backgrounds [95, 42]. These issues do not necessarily terminate the programme closely parallel debates accompanied $f(T)$ gravity—but they must be stated plainly: background-level fits no longer constitute, by themselves, a viability argument for $f(Q)$ cosmology.

6. MATTER–GEOMETRY COUPLING THEORIES

The theories of this section share a defining structural feature: the gravitational Lagrangian depends explicitly on matter variables the matter Lagrangian L_m, the trace $T = g^{\mu\nu}T_{\mu\nu}$, or the squared energy–momentum scalar $\mathbf{T}^2 = T_{\mu\nu}T^{\mu\nu}$—so that in general $\nabla_\mu T^{\mu\nu} \neq 0$. Test particles consequently experience an extra force orthogonal to the four-velocity and move on non-geodesic trajectories [96, 23], and the non-conservation admits a thermodynamic reading as irreversible, gravitationally induced matter creation with an associated creation pressure [31, 32, 33, 34], connecting these models directly to the entropy-production diagnostics of Section 9. This physical distinctiveness is the principal argument for taking the class seriously; the principal argument against is that its perturbative and strong-field structure remains far less examined than that of the trinity extensions, a point to which we return in Section 10.

6.1 $f(R, L_m)$ gravity

For the action $S = \frac{1}{2\kappa^2}\int \sqrt{-g}\, f(R, L_m)\, d^4x$ (a common alternative normalization absorbs κ^2 into f), the field equations are [23]

$$f_R R_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) f_R - \frac{1}{2}(f - f_{L_m}L_m) g_{\mu\nu} = \frac{1}{2} f_{L_m} T_{\mu\nu}, \qquad (17)$$

with non-conservation relation

$$\nabla^\mu T_{\mu\nu} = 2\nabla^\mu \ln f_{L_m} \frac{\partial L_m}{\partial g^{\mu\nu}}. \qquad (18)$$

Equation (18) exhibits a feature absent in GR: the choice of matter Lagrangian ($L_m = -\rho$ versus $L_m = p$ for a perfect fluid, both of which yield the same $T_{\mu\nu}$ in GR) is physically consequential here, altering both the extra force and the cosmological dynamics [97]. Any concrete $f(R, L_m)$ model must therefore declare its choice of L_m explicitly; failure to do so is a recurring source of irreproducibility in the literature. Cosmological applications now span isotropic dark-energy models, anisotropic Bianchi and hypersurface-homogeneous solutions, and two-fluid configurations, with separable models $f(R, L_m) = R/2 + \alpha L_m^n$ constrained against late-time data to give deceleration-to-acceleration transitions at redshifts $z_{tr} \sim 0.5$–0.8, broadly consistent with model-independent reconstructions.

6.2 $f(R, T)$ gravity

With T the trace of the energy–momentum tensor [24],

$$f_R R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) f_R = \kappa^2 T_{\mu\nu} - f_T(T_{\mu\nu} + \Theta_{\mu\nu}), \qquad \Theta_{\mu\nu} \equiv g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}, \qquad (19)$$

where $\Theta_{\mu\nu}$ again depends on the choice of L_m. The minimal model $f = R + 2\lambda T$ has generated an enormous literature of exact isotropic and anisotropic solutions. Two critical points deserve emphasis in any survey of this literature. First, for pressureless dust with $L_m = -\rho$ the minimal model is equivalent, at the background level, to GR with rescaled gravitational and cosmological constants, so that many solutions presented as new physics are disguised general relativity [98]; genuinely new dynamics requires pressure, non-minimal f, or perturbations. Second, the trace coupling explicitly violates energy–momentum conservation, and the associated particle-creation rate must be non-negative for the thermodynamic interpretation to be consistent a constraint on the sign of λ that is frequently ignored but that Section 9 elevates to a selection criterion. Observational analyses of minimal $f(R, T)$ dark energy models find $|\lambda|$ constrained to small values, with the ΛCDM limit statistically preferred by information criteria in most analyses [44].

6.3 $f(Q,T)$ and $f(Q,L_m)$ gravity

Transplanting the matter couplings into the symmetric teleparallel framework yields $f(Q,T)$ gravity [25] and, more recently, $f(Q,L_m)$ gravity [26]. The $f(Q,L_m)$ field equations, in direct parallel with Eq. (17), read

$$\frac{2}{\sqrt{-g}}\nabla_\alpha\left(\sqrt{-g}\, f_Q\, P^\alpha{}_{\mu\nu}\right) + f_Q\left(P_{\mu\alpha\beta}Q_\nu{}^{\alpha\beta} - 2Q^{\alpha\beta}{}_\mu P_{\alpha\beta\nu}\right) + \frac{1}{2} f\, g_{\mu\nu} = \frac{1}{2} f_{L_m}\left(g_{\mu\nu}L_m - T_{\mu\nu}\right), \quad (20)$$

where $P^\alpha{}_{\mu\nu}$ is the non-metricity conjugate, together with a non-conservation relation structurally analogous to Eq. (18). For separable models, $f(Q,L_m) = \alpha Q + \beta L_m^n$ or $f = \alpha Q + \beta g(L_m)$, the FLRW dynamics reduces to algebraic relations between H^2 and ρ whose solutions realize quintessence-like effective equations of state crossing to acceleration at $z_{tr} \simeq 0.5$–0.8, dynamical extrema of the Hubble parameter absent in ΛCDM, and distinctive entropy-production histories. These models inherit the second-order character of $f(Q)$ and, it must be said, its connection-branch subtleties and open perturbative questions. The $f(Q,T)$ branch has been studied both at background level and, partially, at perturbative level [25, 99], while for $f(Q,L_m)$ a systematic perturbation analysis is, to our knowledge, still lacking; we flag this as among the most important open problems in the subfield, since without it the class cannot be confronted with growth data.

6.4 Energy momentum squared gravity $f(R, \mathbf{T}^2)$

Energy–momentum-squared gravity (EMSG) adds the scalar $\mathbf{T}^2 = T_{\mu\nu}T^{\mu\nu}$ to the gravitational Lagrangian [27, 28, 29]. The minimal model $f = R + \eta\, \mathbf{T}^2$ leaves vacuum solutions untouched but modifies high-density regimes: for suitable sign of η the early-universe singularity is replaced by a bounce at finite density [28], neutron-star structure is altered giving the theory rare dual cosmology/astrophysics testing ground and power-law generalizations $f = R + \eta\,(\mathbf{T}^2)^n$ produce late-time acceleration for $n < 1/2$ branches [29, 30]. Constraints on η have been derived from big-bang nucleosynthesis, from neutron-star mass–radius data, and from late-time cosmological observations; published bounds employ differing sign and normalization conventions for η, and we deliberately refrain from quoting a single numerical range here, referring the reader to the primary analyses.

7. CONFRONTATION WITH OBSERVATIONS

7.1 Local and astrophysical tests

Solar-system dynamics constrain the parametrized post-Newtonian parameter γ to $|\gamma - 1| \lesssim 2.3 \times 10^{-5}$ from the Cassini Shapiro-delay measurement, together with tight bounds on Yukawa-type fifth forces and on the time variation of G [69]. Unscreened light scalars coupled to matter with gravitational strength are excluded; every viable curvature-based dark energy model therefore relies on a screening mechanism chameleon [68], symmetron, or Vainshtein [78] and laboratory and astrophysical tests of screening (atom interferometry, dwarf-galaxy morphology, stellar structure) have matured into a discriminating industry [70, 44]. Background-mimicking $f(T)$ and $f(Q)$ models evade local tests trivially at the price of near-degeneracy with GR in that regime, while matter–geometry coupling theories face a qualitatively different local constraint: the extra force on test particles and the non-geodesic motion it induces must be suppressed in the solar system, which for separable models constrains the coupling sector directly.

7.2 The multimessenger constraint

The near-simultaneous detection of GW170817 and GRB 170817A [45, 46] bounds the tensor propagation speed,

$$\left|\frac{c_T}{c} - 1\right| \lesssim 10^{-15}, \quad (21)$$

with immediate structural consequences [47, 48, 49]: quartic and quintic Horndeski sectors are eliminated as dark energy; late-time Gauss–Bonnet couplings are severely restricted; while $f(R)$, $f(T)$, $f(Q)$ and the matter–geometry coupling theories, all of which propagate tensor modes at $c_T = c$ on FLRW backgrounds, pass automatically. This single measurement thus inverted the pre-2017 hierarchy of theoretical interest: the surviving theory space is dominated by precisely the classes emphasized in this review, and the discriminating burden shifts from wave propagation to structure growth and thermodynamics.

7.3 Background cosmological data and methodological cautions

The standard late-time pipeline confronts models with Type Ia supernovae (the Pantheon and Pantheon+ compilations [100, 101]), cosmic chronometer determinations of $H(z)$ [102], baryon acoustic oscillations (e BOSS [103] and now DESI [9]), and CMB distance priors [5], explored with Markov-chain Monte Carlo methods and compared through information criteria.

Table 1 summarizes the benchmark models of each class and the qualitative outcome of such analyses. Rather than tabulating numerical best-fit values which are tied to specific data combinations and priors and rapidly become stale we emphasize two methodological cautions that any reader of this literature should apply. First, analyses that fit an ad hoc parametrization of $H(z)$ or of the scale factor and subsequently reconstruct the gravitational Lagrangian constrain the parametrization, not the theory; only dynamical integration of the field equations from the model parameters yields a genuine test. Second, information-criterion comparisons between papers are meaningful only when data sets, likelihoods and priors coincide; cross-paper AIC tables should be read as indicative at best.

Table 1. Expanded viability scorecard for each modified-gravity class. "Order" = order of metric field equations; "DOF" = extra propagating modes beyond GR; GW = passes GW170817 tensor-speed bound; Solar = compatible with solar-system tests; H_0/S_8 = capacity to ease the respective tension; Thermo = GSL / entropy production analysed. = satisfied; ▲ = partial or model-dependent; × = fails. ^{ch}: screened via chameleon mechanism. ^{c}: $c_T = c$ enforced by construction post-GW170817.

Class	Benchmark	Order	DOF	GW	Solar	H_0	S_8	Thermo
$f(R)$	Hu–Sawicki	4th	1 scalar		^{ch}	▲	▲	
Horndeski	G_2-G_3-G_4	2nd	1 scalar	^{c}	▲	▲	▲	▲
$f(\mathcal{G})$	$R + f(\mathcal{G})$	4th	▲	▲	▲	▲	▲	▲
$f(T)$	$T + \alpha T^b$	2nd	debated			▲	▲	
$f(Q)$	$Q + \alpha Q^n$	2nd	debated			▲		▲
$f(R,T)$	$R + 2\lambda T$	4th	1 scalar		▲	▲	▲	
$f(R,L_m)$	$R + \beta L_m^n$	4th	1 scalar		▲	▲	▲	
$f(Q,L_m)$	$Q + \beta L_m^n$	2nd	debated		▲	▲		
EMSG	$R + \eta \mathbf{T}^2$	4th	▲			▲	▲	▲

Table 2. Comparison of ΛCDM with modified gravity as a class.

Feature	ΛCDM	Modified Gravity
Dark energy origin	Cosmological constant Λ	Infrared modification of geometry
Fine-tuning	$\rho_\Lambda \sim 10^{-47}$ GeV^4	New function f; free parameters
Extra propagating fields	None	Scalaron, connection modes (theory-dep.)
GW speed	$c_T = c$	Usually $c_T = c$ (post-GW170817)
Energy–momentum cons.	$\nabla_\mu T^{\mu\nu} = 0$	Violated in coupling theories
Screening mechanism	Not required	Required for viable curvature theories
Entropy / thermodynamics	Bekenstein–Hawking	Modified Wald entropy; GSL deformed
H_0 tension	Unresolved	Partial (requires early-time action)
S_8 tension	Unresolved	Eased in suppressed-growth $f(Q)$ models
Observational status	Excellent	Actively constrained; perturbations open

7.4 Structure growth

Modified gravity generically alters the linear growth equation

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G_{\rm eff}(a,k)\,\rho_m \delta_m = 0 \tag{22}$$

through an effective coupling $G_{\rm eff}(a,k)$ and a gravitational slip $\eta(a,k) = \Phi/\Psi$, shifting the growth index from its GR value $\gamma \simeq 0.55$ (e.g. $\gamma \simeq 0.68$ for self-accelerating DGP). Redshift-space distortion compilations of $f\sigma_8$ already discriminate between enhanced-growth models such as $f(R)$ with large $|f_{R0}|$ and suppressed-growth models such as certain $f(Q)$ scenarios [104, 92, 93], and the mild preference of the $f\sigma_8$ data for growth weaker than Planck-normalized ΛCDM is precisely the direction the S_8 tension points. The forthcoming full-survey DESI, Euclid and Rubin data will measure $G_{\rm eff}$ and slip at the few-percent level across $0 < z \lesssim 2$ [43], which will be decisive for every class in Table 1 whose perturbation theory is under control a qualification that presently excludes much of the matter–geometry coupling sector and motivates the perturbative programme advocated in Section 12.

8. MODIFIED GRAVITY AND THE COSMOLOGICAL TENSIONS

A candid assessment is required here, because this literature is prone to overclaiming. The H_0 tension [4, 6, 8] is primarily a mismatch between the sound-horizon-calibrated and distance-ladder-calibrated distance scales. Purely late-time modifications of $H(z)$ which describes most of the models in Sections 3–6 are strongly constrained by the inverse distance ladder: raising H_0 while preserving the CMB acoustic scale and the BAO+SNe distance relation requires either reducing the sound horizon at recombination or introducing sharp late-time features, and generic smooth late-time models cannot fully resolve the tension without degrading the joint fit [7]. Modified-gravity scenarios that act at recombination early-dark-energy-like limits or an enhanced early $G_{\rm eff}$ fare better on H_0 but tend to worsen the S_8 discrepancy. Conversely, several $f(Q)$ and related models predict $G_{\rm eff} < G$ at late times and hence suppressed growth [92, 93], which is the direction favoured by the S_8 data but does nothing for H_0. The honest summary we defend is the following: no modified gravity model currently resolves both tensions simultaneously without new fine-tuning; but the tensions define sharp design targets raising H_0 requires early-time action, lowering S_8 requires suppressed late-time growth and the non-metricity and coupling classes naturally supply the latter ingredient. If the DESI indications of an evolving dark-energy equation of state consolidate [9], the case for a genuinely dynamical infrared sector, whether fluid or geometric, will strengthen considerably; the theories reviewed here provide the geometric realizations against which that hypothesis can be tested.

9. THERMODYNAMIC VIABILITY AND THE GENERALIZED SECOND LAW

9.1 The gravity–thermodynamics correspondence

Black-hole thermodynamics [56, 57] and its de Sitter extension [58] suggest that horizon area, entropy and temperature are features of gravitation itself. Jacobson’s derivation of the Einstein equations as an equation of state from the Clausius relation $\delta Q = T\,dS$ on local Rindler horizons [51], the derivation of the Friedmann equations from the first law at the apparent horizon [53, 54], and the emergent-gravity programme [52] together motivate treating horizon thermodynamics as a consistency requirement on gravitational dynamics. For modified theories the correspondence generalizes non-trivially: the horizon entropy is no longer the Beckenstein Hawking area law but the Wald entropy $S_h = Af_R/4G$ in $f(R)$ gravity and for $f(R)$ the Clausius relation acquires an irreversible internal-production term, $dS = \delta Q/T + d_iS$, characteristic of non-equilibrium thermodynamics [55]. Analogous effective entropies hold in $f(T)$ and $f(Q)$ gravity [59, 42].

9.2 The generalized second law at the apparent horizon

For spatially flat FLRW the apparent horizon coincides with the Hubble horizon, $\tilde{r}_A = 1/H$, with area $A = 4\pi/H^2$, temperature

$$T_h = \frac{H}{2\pi}\left(1 - \frac{\dot{H}}{2H^2}\right) \simeq \frac{H}{2\pi}, \tag{23}$$

and horizon entropy $S_h = A/(4G_{\rm eff})$ with the theory-appropriate effective coupling. The matter entropy inside the horizon follows from the Gibbs relation

$$T_m\,dS_m = d(\rho V) + p\,dV, \qquad V = \frac{4}{3}\pi\tilde{r}_A^3. \tag{24}$$

The generalized second law requires

$$\dot{S}_{\rm tot} = \dot{S}_h + \dot{S}_m + \dot{S}_{\rm prod} \geq 0, \tag{25}$$

where $\dot{S}_{\rm prod}$ is the irreversible production term, present whenever the theory sources non-conservation of $T_{\mu\nu}$. In GR with a perfect fluid the GSL holds at the apparent horizon for equations of state $w \geq -1$ and is threatened in phantom regimes; in modified gravity each of the three terms is deformed, and the sign of their sum along a given expansion history is a non-trivial, model-dependent statement.

9.3 Entropy production in matter–geometry coupling theories

For the coupling theories of Section 6 the non-conservation relation implies a source term in the continuity equation,

$$\dot{\rho} + 3H(\rho + p) = \Gamma(t), \tag{26}$$

which the open-system cosmology of Prigogine and collaborators [31, 32] interprets as adiabatic particle creation at rate Γ_n per particle, $\dot{n} + 3Hn = n\,\Gamma_n$, with associated creation pressure

$$p_c = -\frac{\rho+p}{3H}\,\Gamma_n\,, \tag{27}$$

and entropy production

$$\dot{S}_{\text{prod}} = S\,\Gamma_n \geq 0 \Leftrightarrow \Gamma_n \geq 0. \tag{28}$$

Equation (28) is the crux: the second law demands that gravity create matter, not destroy it [31, 33, 34]. For a given $f(R,T)$, $f(R,L_m)$ or $f(Q,L_m)$ model the sign of Γ along the cosmic history is computable directly from the field equations, and parameter ranges that fit background data perfectly well can violate $\Gamma \geq 0$ in the matter era or induce $\dot{S}_{\text{tot}} < 0$ near super-accelerating phases. Thermodynamic viability is therefore a genuine filter, logically independent of the background fit, and because the entropy budget differs between models whose $H(z)$ histories are degenerate a discriminating one. Recent analyses of entropy production and the GSL in $f(Q,L_m)$ cosmologies illustrate the programme concretely, identifying dynamical Hubble extrema and their entropy signatures.

We advocate, as a general methodological conclusion of this review, that $\dot{S}_{\text{tot}}(z) \geq 0$ and $\Gamma(z) \geq 0$ diagnostics computed by dynamical integration of the field equations, not from fitted parametrizations be reported alongside information criteria in observational analyses of matter geometry coupling models.

10. THEORETICAL PATHOLOGIES: A CRITICAL ASSESSMENT

Every class surveyed above carries theoretical liabilities that are as constraining as any data set, and a review that catalogues models without cataloguing pathologies misleads. We organize them by type.

Ostrogradsky instabilities. Higher-derivative theories generically propagate ghosts. $f(R)$ gravity evades the theorem through its degenerate structure the single extra mode is the healthy scalaron provided $f_R > 0$, $f_{RR} > 0$ [36] and DHOST theories evade it by constructed degeneracy. Generic $f(R,\mathcal{G})$ models do not, and their ghost-free windows are narrow [81].

Local Lorentz violation and degrees of freedom. Pure-tetrad $f(T)$ gravity violates local Lorentz invariance [85]; the covariant reformulation [87] shifts the problem to the determination of the spin connection. Hamiltonian counts of the extra degrees of freedom disagree [86, 41], and the reappearance of strongly coupled modes around FLRW undermines linear perturbation theory precisely where cosmological predictions are extracted.

Connection branches and strong coupling in $f(Q)$. The coincident gauge is not universally admissible; inequivalent flat connections yield inequivalent cosmologies, and generic $f(Q)$ exhibits ghostly or strongly coupled perturbative sectors around cosmological backgrounds [42, 95]. Model-building that ignores the connection sector is, at best, incomplete.

The matter-Lagrangian degeneracy. In all theories with explicit L_m or $\Theta_{\mu\nu}$ dependence, physically equivalent choices of L_m in GR become inequivalent, and published results are frequently silent about the choice made. This is a reproducibility problem the subfield should resolve by convention.

Cauchy problem and strong-field structure. Well-posedness of the initial-value problem is established for $f(R)$ under the standard conditions, but remains essentially unexamined for the matter geometry coupling theories; their compact-object phenomenology (with the partial exception of EMSG) rests on static solutions whose dynamical stability is unknown.

Degeneracy with ΛCDM. Finally, a sociological pathology: model families flexible enough to mimic ΛCDM arbitrarily well are, by the same token, difficult to falsify. The scientific value of any modified gravity proposal lies in its rigid predictions the scale-dependent growth of $f(R)$, the suppressed G_{eff} of specific $f(Q)$ models, the creation-rate sign condition of coupling theories and we have organized this review to foreground exactly those.

11. OPEN PROBLEMS AND FUTURE DIRECTIONS

Several problems of both theoretical and observational character remain unresolved and define the research programme of the coming decade.

Perturbative structure of coupling theories. The most urgent open problem is a systematic cosmological perturbation analysis of $f(R,L_m)$ and $f(Q,L_m)$ gravity, including the connection sector in the non-metricity case. Without ghost and strong-coupling analyses, claims about suppressed growth or enhanced entropy production at the linear level are built on foundations whose stability is untested. This is, at present, a serious reproducibility gap.

Euclid, Rubin, and the Roman Space Telescope. The Euclid satellite (launched 2023), the Rubin Observatory LSST (first light 2025), and the Nancy Grace Roman Space Telescope will together map the three-dimensional distribution of galaxies and the weak gravitational lensing shear field over billions of light-years. For modified gravity they deliver sub-percent measurements of $G_{\text{eff}}(a,k)$, the gravitational slip η, and the growth index γ across redshifts $0 \lesssim z \lesssim 2$ sufficient to falsify every class in Table 1 that predicts order-unity departures from GR in structure growth.

LISA, Einstein Telescope, and Cosmic Explorer. The forthcoming GW detector network tests gravity in the strong-field, radiative regime. For modified gravity the decisive observables are: the tensor-mode luminosity distance $d_L^{GW}(z)$ versus the electromagnetic $d_L^{EM}(z)$, which a running effective Planck mass splits; the GW polarization content; and the quasinormal-mode spectrum of black-hole ringdowns (black-hole spectroscopy). None of these is accessible at current detector sensitivity.

Machine learning and Bayesian model selection. A rapidly growing frontier applies neural posterior estimation, simulation-based inference, and normalizing flows to the simultaneous confrontation of many modified-gravity models against multi-probe data. These methods bypass the need for analytic likelihoods and enable rigorous Bayesian model comparison across the full prior volume. Their application to $f(Q, L_m)$ and EMSG is, as of this writing, essentially unexplored.

Quantum gravity connections. Modified gravity theories are classical; the question of which effective field theory structure, if any, they represent at the quantum level is largely open. The non-minimal matter–geometry coupling of the $f(R, L_m)$ and $f(Q, L_m)$ type, in particular, lacks a known ultraviolet completion. Understanding the quantum origin and stability of these theories remains a fundamental challenge connecting modified gravity to quantum gravity research.

12. SUMMARY AND OUTLOOK

Three conclusions organize the survey. First, GW170817 inverted the field: the theory space that survives trivially $f(R)$, $f(T)$, $f(Q)$ and the matter–geometry couplings is precisely the space where background degeneracy with ΛCDM is easiest, so that discrimination must come from perturbations, from screening tests, and from thermodynamics. Second, the matter–geometry coupling theories have matured from exact-solution mining into a framework with distinctive, testable physics creation pressure, non-geodesic motion, entropy production but their perturbative health is largely unexamined; we regard a systematic cosmological perturbation analysis of $f(R, L_m)$ and $f(Q, L_m)$ gravity, including the connection sector in the latter, as the single most important open theoretical problem in the subfield. Third, the observational decade ahead the full DESI survey, Euclid, Rubin LSST, and gravitational-wave cosmology will shrink the surviving space dramatically. Standard sirens will measure H_0 independently of both the distance ladder and the sound horizon, arbitrating the tension that motivates much of this activity; and in theories with a running effective Planck mass the gravitational-wave luminosity distance departs from the electromagnetic one, $d_L^{GW}(z) \neq d_L^{EM}(z)$, a clean null test of GR that Einstein Telescope-class detectors will perform to percent precision [105]. Models should henceforth be published with falsifiable perturbative and thermodynamic predictions, not background fits alone; where this review has a single message, it is that one.

A brief note on the growing role of machine learning and artificial intelligence in observational cosmology is warranted. Neural-network emulators of Boltzmann codes, likelihood-free inference pipelines, and deep-learning classifiers for photometric redshifts are already embedded in the analysis chains of Euclid and Rubin. For modified gravity, simulation-based inference enables the sampling of parameter spaces in which traditional MCMC is computationally prohibitive, and graph neural networks applied to the cosmic web encode information beyond the power spectrum that is sensitive to screening and to the extra force of coupling theories. This algorithmic revolution does not change the underlying physics the field equations remain the arbiters of viability but it dramatically accelerates the confrontation of theory with data and will be central to the programme of the next decade.

List of Abbreviations

AIC	Akaike Information Criterion
BAO	Baryon Acoustic Oscillations
BBH	Binary Black Hole
BIC	Bayesian Information Criterion
BNS	Binary Neutron Star
CC	Cosmic Chronometers

CMB	Cosmic Microwave Background
DESI	Dark Energy Spectroscopic Instrument
EMSG	Energy–Momentum-Squared Gravity
FLRW	Friedmann–Lemaître–Robertson–Walker
GR	General Relativity
GSL	Generalized Second Law
GW	Gravitational Wave
ISW	Integrated Sachs–Wolfe effect
LSST	Legacy Survey of Space and Time (Rubin)
MCMC	Markov-Chain Monte Carlo
OHD	Observational Hubble Data (cosmic chronometers)
PPN	Parametrized Post-Newtonian
PTA	Pulsar Timing Array
SNe	Type Ia Supernovae
STEGR	Symmetric Teleparallel Equivalent of GR
TEGR	Teleparallel Equivalent of GR
TT	Transverse–Traceless

ACKNOWLEDGEMENTS

This review paper has been prepared by S. R. Bhoyar (Supervisor) and A. D. Wankhade (Research Scholar) in partial fulfillment of the requirements for the Ph.D. coursework prescribed by Sant Gadge Baba Amravati University, Amravati, Maharashtra, India, through the research center, Department of Mathematics, Phulsing Naik Mahavidyalaya, Pusad-445216, Maharashtra, India.

REFERENCES

[1] A. G. Riess et al., Astron. J. 116 (1998) 1009.

[2] S. Perlmutter et al., Astrophys. J. 517 (1999) 565.

[3] S. Weinberg, Rev. Mod. Phys. 61 (1989) 1.

[4] A. G. Riess et al., Astrophys. J. Lett. 934 (2022) L7.

[5] N. Aghanim et al. [Planck Collaboration], Astron. Astrophys. 641 (2020) A6.

[6] E. Di Valentino et al., Class. Quantum Grav. 38 (2021) 153001.

[7] L. Perivolaropoulos and F. Skara, New Astron. Rev. 95 (2022) 101659.

[8] E. Abdalla et al., J. High Energy Astrophys. 34 (2022) 49.

[9] A. G. Adame et al. [DESI Collaboration], arXiv:2404.03002.

[10] P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. 75 (2003) 559.

[11] E. J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. D 15 (2006) 1753.

[12] J. A. Frieman, M. S. Turner and D. Huterer, Ann. Rev. Astron. Astrophys. 46 (2008) 385.

[13] V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D 9 (2000) 373.

[14] M. Li, Phys. Lett. B 603 (2004) 1.

[15] S. Wang, Y. Wang and M. Li, Phys. Rept. 696 (2017) 1.

[16] D. Lovelock, J. Math. Phys. 12 (1971) 498.

[17] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, Phys. Rept. 513 (2012) 1.

[18] A. Joyce, B. Jain, J. Khoury and M. Trodden, Phys. Rept. 568 (2015) 1.

[19] K. Koyama, Rept. Prog. Phys. 79 (2016) 046902.

[20] K. Bamba, S. Capozziello, S. Nojiri and S. D. Odintsov, Astrophys. Space Sci. 342 (2012) 155.

[21] J. Beltrán Jiménez, L. Heisenberg and T. Koivisto, Phys. Rev. D 98 (2018) 044048.

[22] J. Beltrán Jiménez, L. Heisenberg and T. Koivisto, Universe 5 (2019) 173.

[23] T. Harko and F. S. N. Lobo, Eur. Phys. J. C 70 (2010) 373.

[24] T. Harko, F. S. N. Lobo, S. Nojiri and S. D. Odintsov, Phys. Rev. D 84 (2011) 024020.

[25] Y. Xu, G. Li, T. Harko and S.-D. Liang, Eur. Phys. J. C 79 (2019) 708.

[26] A. Hazarika, S. Arora, P. K. Sahoo and T. Harko, arXiv:2407.xxxxx.

[27] N. Katırcı and M. Kavuk, Eur. Phys. J. C 74 (2014) 2812.

[28] M. Roshan and F. Shojai, Phys. Rev. D 94 (2016) 044002.

[29] C. V. R. Board and J. D. Barrow, Phys. Rev. D 96 (2017) 123517.

[30] O. Akarsu, N. Katirci and S. Kumar, Phys. Rev. D 97 (2018) 024011.

[31] I. Prigogine, J. Geheniau, E. Gunzig and P. Nardone, Gen. Rel. Grav. 21 (1989) 767.

[32] M. O. Calvão, J. A. S. Lima and I. Waga, Phys. Lett. A 162 (1992) 223.

[33] T. Harko, Phys. Rev. D 90 (2014) 044067.

[34] T. Harko, F. S. N. Lobo, J. P. Mimoso and D. Pavón, Eur. Phys. J. C 75 (2015) 386.

[35] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82 (2010) 451.

[36] A. De Felice and S. Tsujikawa, Living Rev. Rel. 13 (2010) 3.

[37] S. Capozziello and M. De Laurentis, Phys. Rept. 509 (2011) 167.

[38] S. Nojiri and S. D. Odintsov, Phys. Rept. 505 (2011) 59.

[39] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Phys. Rept. 692 (2017) 1.

[40] Y.-F. Cai, S. Capozziello, M. De Laurentis and E. N. Saridakis, Rept. Prog. Phys. 79 (2016) 106901.

[41] S. Bahamonde et al., Rept. Prog. Phys. 86 (2023) 026901.

[42] L. Heisenberg, Phys. Rept. 1066 (2024) 1.

[43] M. Ishak, Living Rev. Rel. 22 (2019) 1.

[44] E. N. Saridakis et al. [CANTATA Collaboration], Modified Gravity and Cosmology: An Update by the CANTATA Network, Springer (2021), arXiv:2105.12582.

[45] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. 119 (2017) 161101.

[46] B. P. Abbott et al., Astrophys. J. Lett. 848 (2017) L13.

[47] J. M. Ezquiaga and M. Zumalacárregui, Phys. Rev. Lett. 119 (2017) 251304.

[48] P. Creminelli and F. Vernizzi, Phys. Rev. Lett. 119 (2017) 251302.

[49] T. Baker, E. Bellini, P. G. Ferreira, M. Lagos, J. Noller and I. Sawicki, Phys. Rev. Lett. 119 (2017) 251301.

[50] P. Creminelli, M. Lewandowski, G. Tambalo and F. Vernizzi, JCAP 1812 (2018) 025.

[51] T. Jacobson, Phys. Rev. Lett. 75 (1995) 1260.

[52] T. Padmanabhan, Rept. Prog. Phys. 73 (2010) 046901.

[53] R.-G. Cai and S. P. Kim, JHEP 0502 (2005) 050.

[54] M. Akbar and R.-G. Cai, Phys. Lett. B 635 (2006) 7.

[55] C. Eling, R. Guedens and T. Jacobson, Phys. Rev. Lett. 96 (2006) 121301.

[56] J. D. Bekenstein, Phys. Rev. D 7 (1973) 2333.

[57] S. W. Hawking, Commun. Math. Phys. 43 (1975) 199.

[58] G. W. Gibbons and S. W. Hawking, Phys. Rev. D 15 (1977) 2738.

[59] K. Bamba and C.-Q. Geng, JCAP 1111 (2011) 008.

[60] H. A. Buchdahl, Mon. Not. Roy. Astron. Soc. 150 (1970) 1.

[61] A. A. Starobinsky, Phys. Lett. B 91 (1980) 99.

[62] W. Hu and I. Sawicki, Phys. Rev. D 76 (2007) 064004.

[63] A. A. Starobinsky, JETP Lett. 86 (2007) 157.

[64] S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, Phys. Rev. D 70 (2004) 043528.

[65] T. Chiba, Phys. Lett. B 575 (2003) 1.

[66] L. Amendola, R. Gannouji, D. Polarski and S. Tsujikawa, Phys. Rev. D 75 (2007) 083504.

[67] Y.-S. Song, W. Hu and I. Sawicki, Phys. Rev. D 75 (2007) 044004.

[68] J. Khoury and A. Weltman, Phys. Rev. Lett. 93 (2004) 171104.

[69] C. M. Will, Living Rev. Rel. 17 (2014) 4.

[70] C. Burrage and J. Sakstein, Living Rev. Rel. 21 (2018) 1.

[71] G. W. Horndeski, Int. J. Theor. Phys. 10 (1974) 363.

[72] C. Brans and R. H. Dicke, Phys. Rev. 124 (1961) 925.

[73] B. Ratra and P. J. E. Peebles, Phys. Rev. D 37 (1988) 3406.

[74] A. Nicolis, R. Rattazzi and E. Trincherini, Phys. Rev. D 79 (2009) 064036.

[75] C. Deffayet, G. Esposito-Farese and A. Vikman, Phys. Rev. D 79 (2009) 084003.

[76] R. Kase and S. Tsujikawa, Int. J. Mod. Phys. D 28 (2019) 1942005.

[77] G. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B 485 (2000) 208.

[78] A. I. Vainshtein, Phys. Lett. B 39 (1972) 393.

[79] S. Nojiri and S. D. Odintsov, Phys. Lett. B 631 (2005) 1.

[80] S. D. Odintsov and V. K. Oikonomou, Phys. Lett. B 797 (2019) 134874.

[81] A. De Felice and S. Tsujikawa, Phys. Lett. B 675 (2009) 1.

[82] R. Ferraro and F. Fiorini, Phys. Rev. D 75 (2007) 084031.

[83] G. R. Bengochea and R. Ferraro, Phys. Rev. D 79 (2009) 124019.

[84] E. V. Linder, Phys. Rev. D 81 (2010) 127301.

[85] B. Li, T. P. Sotiriou and J. D. Barrow, Phys. Rev. D 83 (2011) 064035.

[86] T. P. Sotiriou, B. Li and J. D. Barrow, Phys. Rev. D 83 (2011) 104030.

[87] M. Krššák and E. N. Saridakis, Class. Quantum Grav. 33 (2016) 115009.

[88] S. Bahamonde, C. G. Böhmer and M. Wright, Phys. Rev. D 92 (2015) 104042.

[89] J. Beltrán Jiménez, L. Heisenberg, T. Koivisto and S. Pekar, Phys. Rev. D 101 (2020) 103507.

[90] R. Lazkoz, F. S. N. Lobo, M. Ortíz-Baños and V. Salzano, Phys. Rev. D 100 (2019) 104027.

[91] F. K. Anagnostopoulos, S. Basilakos and E. N. Saridakis, Phys. Lett. B 822 (2021) 136634.

[92] B. J. Barros, T. Barreiro, T. Koivisto and N. J. Nunes, Phys. Dark Univ. 30 (2020) 100616.

[93] N. Frusciante, Phys. Rev. D 103 (2021) 044021.

[94] F. D'Ambrosio, S. D. B. Fell, L. Heisenberg and S. Kuhn, Phys. Rev. D 105 (2022) 024042.

[95] D. A. Gomes, J. Beltrán Jiménez, A. Jiménez Cano and T. S. Koivisto, Phys. Rev. Lett. 132 (2024) 141401.

[96] O. Bertolami, C. G. Böhmer, T. Harko and F. S. N. Lobo, Phys. Rev. D 75 (2007) 104016.

[97] O. Bertolami, F. S. N. Lobo and J. Páramos, Phys. Rev. D 78 (2008) 064036.

[98] S. B. Fisher and E. D. Carlson, Phys. Rev. D 100 (2019) 064059.

[99] A. Nájera and A. Fajardo, JCAP 2203 (2022) 020.

[100] D. M. Scolnic et al., Astrophys. J. 859 (2018) 101.

[101] D. Brout et al., Astrophys. J. 938 (2022) 110.

[102] M. Moresco et al., Living Rev. Rel. 25 (2022) 6.

[103] S. Alam et al. [eBOSS Collaboration], Phys. Rev. D 103 (2021) 083533.

[104] S. Nesseris, G. Pantazis and L. Perivolaropoulos, Phys. Rev. D 96 (2017) 023542.

[105] E. Belgacem, Y. Dirian, S. Foffa and M. Maggiore, Phys. Rev. D 97 (2018) 104066.